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Length of the Lunar Crescent

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SUMMARY

The thin crescent Moon is observed not to extend the full 180° arc expected if the Moon had an ideal surface. I report on a collection of 65 crescent arc length observations made across North America on 1989 April 6. These observations show the crescent length to be independent of whether or not the observer used any optical aid. Since the resolution of a naked eye observer is more than an order of magnitude larger then either the seeing disk size or the resolution of a telescopic observer, the effect of the seeing on the perceived arc length must be negligible. A theory to explain the crescent shortening is that the brightness falls off steeply to the cusps so that visual observers will not detect the cusps because they are fainter than the threshold for visual detection. Lunar surface brightnesses are calculated according to the accurate model by Hapke (1984), and it is found that all Danjon's (1932, 1936) collected observations and my new data are well fitted by this model. One implication for lunar calendars is that the crescent cannot be detected if the Moon is within 7° of the Sun, although other effects will always cause the Moon to be invisible until the distance from the Sun is significantly greater.

1 INTRODUCTION

If the Moon were an ideal sphere, then the crescent should appear to extend from pole to pole, for a total extent of 180° of arc around the centre of the lunar disk. But this is not seen, in that thin crescents extend an arc of considerably less than 180°. Danjon (1932, 1936) presents 75 observations which show that this shortening increases as the Moon gets closer to the Sun.

This effect cannot be caused by shadows of lunar mountains shifting the terminator closer to the Sun because the required shadow length would have to be a function of the Earth's position and because mountain chains would have to average over 12000 m in height. McNally (1983) proves that the arc shortening cannot be caused by distortions of the Moon's shape from sphericity. Instead, he proposes that the shortening is caused by atmospheric turbulence (seeing) which will blur the narrow cusps to invisibility. He points out that the characteristic size of the seeing disk low on the horizon may be 5 arcsec and that 'there would be little point in expecting to see those parts of the lunar crescent with a width less than this value'. He then shows that the invisible parts of the cusp have widths from 2–6 arcsec and less depending on lunar phase. In his model, the dependence on phase would result because crescents close to the Sun must be viewed at low altitude through bright twilight.

Danjon (1936) deduced that when the Moon was 7° from the Sun, the arc length would be 0°. In other words, the Moon must be invisible whenever it is within 7° of the Sun. This limit is widely quoted and discussed within the

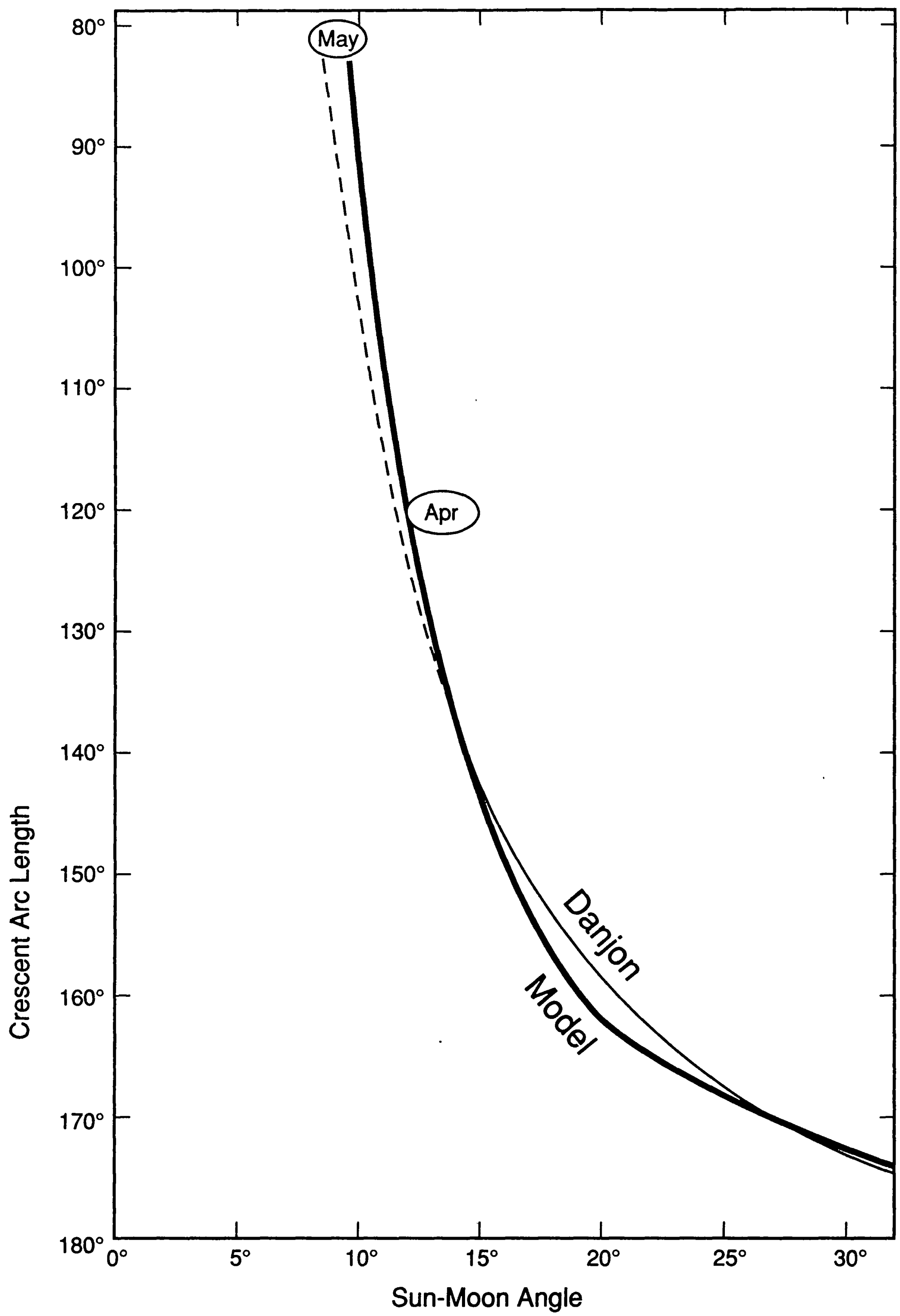

FIG. 1. Arc length shortening as a function of the Sun–Moon angle. The observations are represented by a narrow curve for Danjon's 71 observations with a Sun–Moon angle of greater than 14°, a dashed curve for Danjon's 4 observations with a Sun–Moon angle less than 14°, an ellipse labelled 'Apr' for 35 visual observations on the evening of 1989 April 6, and an ellipse labelled 'May' for 5 observations on the evening of 1989 May 5. The model predictions (see section 4) are given as a thick curve. The agreement between the model and the observations is remarkably good, and provides strong support for the validity of the model.

Islamic community (Ilyas 1984a) as being a criterion for visibility, as being a lower limit on visibility and as being erroneous. The fervour of these discussions arises from the need to regulate the Islamic lunar calendar, where a typical application is the use of Danjon's limit to claim that a particular sighting of the Moon was either possible or impossible.

Doggett & Schaefer (1991) have collected a large number of crescent sightings made across North America during several Moonwatches to provide data for comparison with the various lunar visibility algorithms. One by-product of this database is that many observers reported the arc length. This provides a large database for testing McNally's hypothesis.

2 OBSERVATIONS

Doggett & Schaefer (1989) appealed for amateur astronomers to look for the thin crescent Moon on the evening of 1989 April 6. Across North America, the age of the Moon varied from 20–24 h, the Sun–Moon angle varied from 12°–14°, and the Moon appeared within 3° of azimuth from directly above the Sun. The Sun–Moon angle (the angle separating the Sun and Moon as viewed by an observer on Earth) is the supplement of the phase angle and is also known as the 'arc of light'. A total of 65 observations was reported of the crescent arc length made with either the naked eye, visually but with optical aid, photographically with optics of focal length less than 300 mm, or photographically with optics of focal length greater than 500 mm. All photographic measurements were made by myself using a method similar to that described by Danjon (1932). The observed arc lengths are summarized in Table I & Fig. 1. I find that all photographs from all across America show the same characteristic bright and dark features along the cusp, so that the features must be due to surface albedo and roughness variations on the Moon and not to local observing conditions.

I made a series of arc length estimates on 1989 April 6 from Bowie, Maryland with binoculars. These observations are presented in Table II.

Doggett & Schaefer (1991) and DiCicco (1989) summarize several observations made on the evening of 1989 May 5, when the Moon was 9° away from the Sun. We know of five reports made either through a telescope or photographically with a long focal length lens for which the average arc length was 81° with an rms deviation of 29°. This point is also plotted in Fig. 1.

Danjon (1932, 1936) presents 75 individual observations collected over many lunations by various observers across Europe. These observations were both visual and photographic, with many being precisely measured with a micrometer by transit timings. Danjon notes that all methods of measuring give close agreement on the arc length as a function of the Sun–Moon angle. He calculates the mean value of the observed arc shortening as a function of the Sun–Moon angle. This average function is plotted as a curve in Fig. 1, while the values below 14° are represented by a dashed curve because they are based on only four observations. I note that Danjon's 1931 August 13 observations had a Sun–Moon angle of 10·4°. The rms scatter of his arc length observations for three ranges of Sun-Moon angle are tabulated in Table III.

TABLE I

1989 April 6 Moonwatch observations

Observing Method	Number	Mean Arc Length (degrees)	RMS Deviation (degrees)
Naked eye	12	123	13
With binoculars or telescope	23	117	21
Photographic ($F < 300$ mm)	18	105	12
Photographic ($F > 500$ mm)	12	115	15

TABLE II

1989 April 6 observations from Bowie, Maryland

Universal Time	Circumstances	Crescent Arc Length (degrees)
0:01	first sighted in binoculars	90
0:05	first sighted with naked eye	100
0:10	best visibility	130
0:16	lost to unaided eye	—
0:24	in thick haze layer	80
0:26	lost to binoculars	—

TABLE III

Scatter in observed arc lengths

Source	Sun–Moon angle (degrees)	RMS deviation Observed (degrees)	RMS deviation Predicted (degrees)
Danjon (1932, 1936)	25–30	7	3
Danjon (1932, 1936)	22–23	5	7
Danjon (1932, 1936)	14–15	17	22
1989 April 6 Moonwatch	12–14	19	18
1989 May 5 Moonwatch	9	29	28

The observations are remarkably consistent with each other from lunation to lunation. For example, in Fig. 1 the April and May 1989 values overlap the values found by Danjon. Danjon's data were collected over many lunations, yet his values are all consistent with a normal observational error around his mean curve.

3 ATMOSPHERIC TURBULENCE

McNally (1983) hypothesizes that atmospheric turbulence (seeing) causes the crescent to be invisible where the cusp is narrower than the size of the seeing disk. The idea is that if seeing is large compared to the width, then the cusp will be significantly broadened so that the surface brightness will be smaller than if the seeing were perfect. He claims that the lower surface brightness will render the cusp invisible to the eye. This hypothesis has severe troubles both theoretically and observationally.

The observational task of detecting a portion of the lunar cusp is that of detecting a thin strip of light. Extensive physiological experimentation (e.g.

Lamar *et al.* 1947) has been carried out to test the threshold brightness needed for rectangles of various lengths and widths to be visible. These laboratory experiments test a virtually identical observational task as detecting a portion of the lunar cusp. That is, the width of the perceived rectangle could have been caused by either the actual size of the rectangle or by atmospheric smearing of a narrower rectangle. Their analysis shows that detection of each portion of the rectangle is independent of other sections, so that the nearby brighter and dimmer portions do not affect the visibility of any particular segment. The results of these experiments show decisively that the visibility of long thin rectangles is independent of the width and is dependent on the total light per unit length, provided that the width is smaller than the resolution of the eye. Similarly, the experiments (see also Blackwell 1946) show that unresolved circles of light have a threshold that is independent of the source size, yet which depends on the total brightness within the circle. The basic idea is that all that matters is how much light is received by any resolution element of the eye and not how the light is spread over the 'pixel'. The average resolution of the human eye is 42″ or larger (Blackwell 1946) which is much larger than the relevant crescent widths even with extreme seeing conditions (McNally 1983). In other words, the cusps are always too narrow to be resolved (for any seeing conditions) so that the apparent width of the cusps does not affect visibility. Therefore, the physiological data show that the effects of seeing on the width of the cusp are irrelevant to the problem of the apparent length of the lunar crescent.

For visual observations, the relevant quantity is the width of the image on the retina (I). This width will depend on the width of the crescent at the point of interest (W) as convolved with the size of the seeing disk (S) and the resolution of the human eye (E) with the appropriate magnification (M). To a close approximation, the apparent width will be given by

$$I^2 = (MW)^2 + (MS)^2 + E^2. \qquad (1)$$

The width of the crescent will be a function of the angle from the middle of the crescent (ϕ)

$$W = R(1 - \cos(\theta)/(1 - \sin^2(\phi)\sin^2(\theta))^{0.5}), \qquad (2)$$

where R is the angular radius of the Moon (roughly 0·25° or 900″). The Sun–Moon angle (the angle between the centres of the Sun and Moon as viewed from Earth with no topocentric or refraction corrections) is θ. The resolution of the human eye is typically 42″ for the relevant illumination conditions (Blackwell 1946). McNally adopts a reasonable value for S near the horizon of 5″.

For naked eye observations ($M = 1$), the E contribution will always greatly dominate the S contribution to I because $E \gg S$. The E contribution in equation 1 will also dominate the W contribution for the cusp region. When the E term dominates over both the S and W terms, I will be close to E. In other words, for naked eye observations, the perceived width of the cusps will be close to the resolution of the eye and will have no dependence on the size of the seeing disk. So the eye will perceive the crescent through very good and very bad turbulence in an identical way. Therefore, the seeing is irrelevant to the visibility of the cusps for unaided vision.

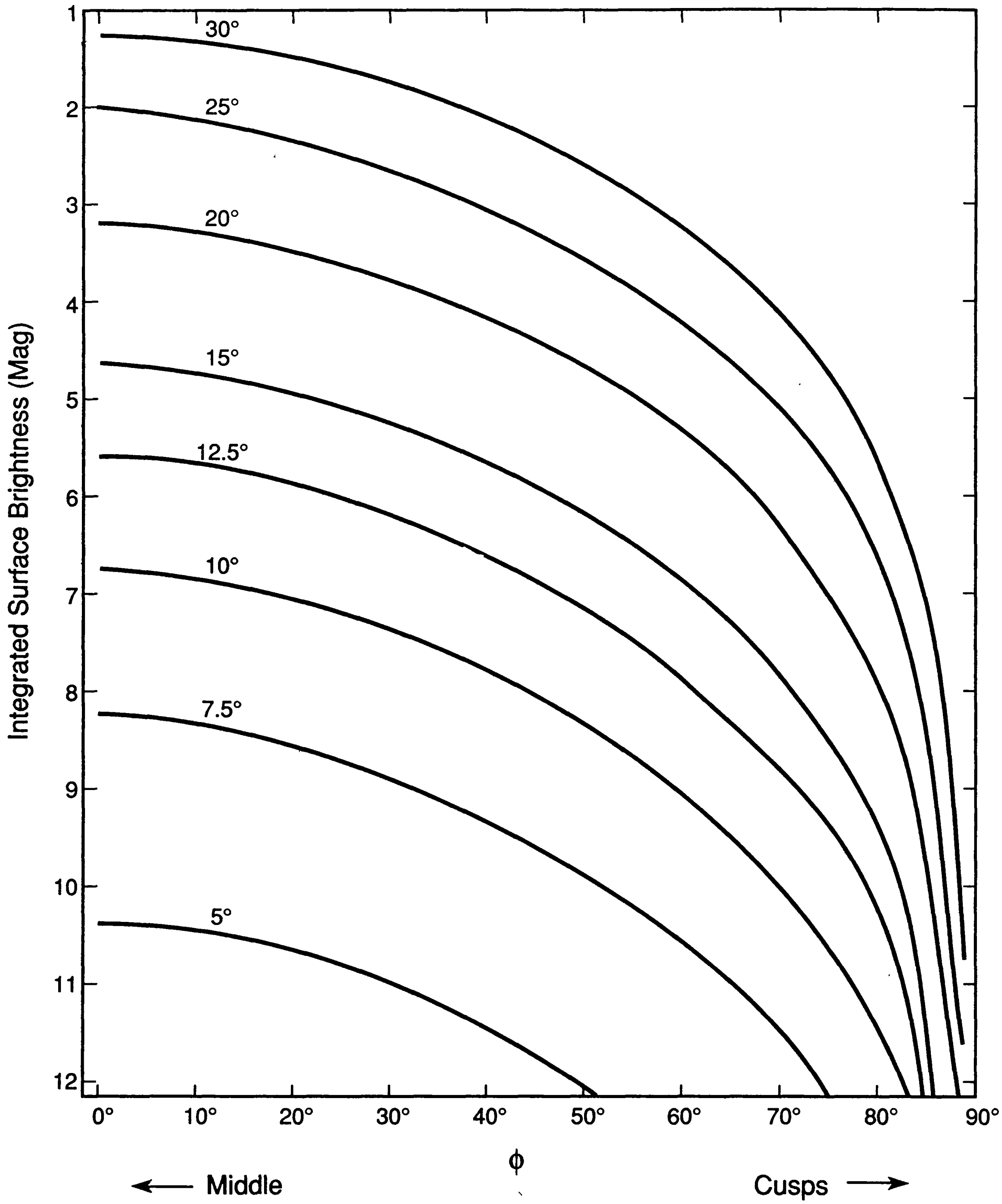

FIG. 2. Brightness across the crescent. The plotted curves are the surface brightness of the Moon integrated along a lunar radius as a function of the angle ϕ from the centre of the crescent towards the cusps. Each curve is for a different Sun–Moon angle. The brightness per unit length falls off quickly as the cusp is approached. The threshold for detection by the human eye is roughly 8·0 magnitudes, with a rms variation of 1·5 magnitudes caused by varying observer sensitivity, etc. When the brightness of the crescent falls below this threshold, the cusp will not be detected. The observed arc length of the crescent will be twice the critical ϕ at which the cusp is too dim to be sighted. The model prediction in Fig. 1 was constructed from this figure. The curves in this figure were constructed using the Hapke equations and the values for the Hapke parameters found by Helfenstein & Veverka (1981).

For telescopic observations (say $M = 100$), the contribution to I from the resolution of the human eye is negligible. That is, E^2 will be much smaller than $(MS)^2$. For the tips of the cusp (where W is near zero), I will be roughly MS or 500″. For the same situation, a naked eye observer would have I equal to roughly 42″. By McNally's hypothesis, the naked eye and telescopic observers should have a grossly different reported arc shortening. Yet the data in Table 1 show that the arc length is not significantly different for unaided and telescopic observers, despite the order of magnitude difference in I. This shows that, in practice, the width of the image on the retina has no apparent effect on the detectability of the crescent. Therefore, the seeing has no significant effect on the arc length even for telescopic observers.

So there are three strong arguments why the seeing cannot cause the arc shortening: firstly, physiological experiments show that the visibility of unresolved sources does not depend on the smearing imposed on the source. Secondly, the resolution of the human eye is always much larger than the size of the seeing disk so that the seeing has no appreciable effect on the perceived width and hence visibility of the cusp. Thirdly, telescopic observers report essentially the same arc shortening as that reported by visual observers, even though the perceived cusp widths differ by large amounts.

The reason why McNally (1983) derived a different equation for Danjon's 'deficiency arc' (McNally's equation 2·1 instead of this equation 2·2) is because he adopted a different definition for the deficiency arc. That is, in his Fig. 2, he constructed the spherical triangle BQZ such that the angle BQZ is a right angle, whereas Danjon chose to construct the triangle with a different point (call it Z′) where QZ′B was the right angle. Danjon's definition provides a meaningful physical interpretation of the arc QZ′ as the shadow of a mountain at Q, whereas McNally's definition for QZ has no ready interpretation.

4 NEW EXPLANATION

McNally has shown that the Moon is sufficiently close to a sphere that there must be light coming from the thin cusps. He was the first to realize that, since the light must be there, the invisibility of the cusps must be caused by some sort of a detection threshold effect.

McNally's particular mechanism involving atmospheric seeing is not viable, but there are several other effects that render the thin cusps difficult to spot. These effects make the total brightness along the crescent fall off steeply as the cusp is approached. The detection threshold for the human eye is roughly some constant, and the parts of the crescent whose brightness is below this threshold will not be visible.

For naked eye observations, the critical portions of the crescent are always narrower than the resolution of the eye. In such a case, the detection threshold does not depend on the surface brightness of the Moon, but on the total brightness integrated across the crescent (Lamar *et al.* 1947).

The brightness integrated across the crescent will fall off sharply towards the cusps for three reasons. The first reason is simply that the crescent gets rapidly narrower. The second reason is that the cusps are regions solely illuminated by the Sun very close to the local horizon and hence on average the polar terrain is illuminated less than equatorial terrain because of greater foreshortening. The third reason is that macroscopic roughness of the Moon's surface creates shadows at the lunar poles that cover up more of the illuminated surface than at the lunar equator.

To analyse this model quantitatively, I need to calculate the surface brightness of the Moon at any part of the illuminated hemisphere. This is a complicated problem with a long history (Minnaert 1961). Hapke (1984) presents an analysis of all the above effects as a series of long and complicated equations. These equations are dependent on six parameters that characterize any surface. For the Moon, Helfenstein & Veverka (1987) review all data and derive the best fitting Hapke parameters, which I have adopted. I have used Hapke's equations to evaluate the lunar surface brightness and have then integrated these surface brightnesses along various lines from the centre of the lunar disk to the limb. This gave the total brightness per unit length of the crescent. I have then converted this brightness into a magnitude by taking 2·5 times the logarithm of the brightness, since the eye is a logarithmic detector. (This is called Fechner's Law, but a power law representation is more precise [Young 1990].) These calculated magnitudes are presented graphically in Fig. 2. The zero of the magnitude scale is arbitrarily chosen.

All else being equal, the threshold for detectability by the human eye will be some constant magnitude (this could be represented as a horizontal line in Fig. 2). The exact value of the threshold has been calculated by a long and intricate calculation: here, however, I will treat this threshold as a free parameter. So my quantitative model predicts that the crescent can be detected only out to the θ value where the magnitude falls below the one free parameter of the threshold. If the threshold magnitude is 8·0, then the predicted arc length as a function of θ closely matches the observations (see Fig. 1).

5. VARIATIONS IN CRESCENT LENGTH

For a given Sun–Moon angle, the measured crescent length will vary from observer to observer and from lunation to lunation. Tables 1 and 3 summarize the observations of the rms scatter of the measured length. Danjon has noted that this variation can be characterized as a random scatter around some stable average value. These variations must be explained by the new model proposed in the last section.

Within the new model, the apparent length of the crescent will be determined by the ϕ value where the brightness curve passes below the threshold. But both the brightness curve and the threshold will be modified by various specific conditions.

The brightness curve will be modified by the presence of local features on the Moon which change from lunation to lunation because of libration. These local features could arise from regional differences of albedo or macroscopic roughness. For example, a mountainous region would have deeper shadows and hence appear darker than would a flat region. Similarly, a flooded crater bottom might have an albedo that is half that of a highland region (Minnaert 1961). The calculated model brightness curves in Fig. 2 are for the disk integrated parameters, so that local variations on the Moon will cause bright and dark segments along the cusp as seen in many crescent photographs. For any particular lunation, all observers should see the same details (as reported for the two Moonwatches), while later lunations will have different details because the libration conditions will not match.

Within the framework of the new model, these variations in local terrain on the Moon will lead to variations in the brightness curve about the average curve as given in Fig. 2. For albedo variations, the brightness from a single spot might vary by up to a factor of two, although the brightness variations integrated across the cusp will be smaller. For roughness variations, the effects are difficult to quantify, but the variations are likely to be comparable to those from albedo. On average, the brightness curve will have variations of roughly a factor of two. A bump in the brightness curve of a factor of 2 is equivalent to a shift in the threshold by a factor of 2, that is to say by 0·7 magnitudes.

The threshold for cusp detection will be roughly constant, yet there are several factors that can slightly change the threshold depending on the particular conditions. These variable shifts in the threshold will arise from complications that result in variations in the observed crescent length.

One complication is that the sensitivity of the human eye varies from observer to observer. In a similar observational task, Schaefer (1990) found that the HWHM variation in the sensitivity of observers is much less than 0·75 magnitude. This would lead to an increased scatter around the model prediction that is small compared with the observed scatter.

Another complication arises because the act of detection is a probabilistic event. Blackwell (1946) presents data that show that the 16% and 84% probability levels are 0·5 magnitudes away from the 50% probability level on average. This will also lead to an increased scatter in the model predictions.

Another complication arises because the human eye has differing sensitivities for detecting a line depending on whether the line is vertical, horizontal, or oblique for the observer. This orientation will vary from sighting to sighting and will be different for the two cusps. The data of Ogilvie & Taylor (1958) with the probability curves of Blackwell (1946) imply that, for random orientations, the rms variation in the threshold will be 0·25 magnitudes due to orientation effects.

Another complication is that, for a given observer on a given night, the extinction will increase as the Moon sets so that the arc length will depend on the time of observation. The effect due to increased absorption can be easily estimated. For the Moon close to the Sun, the altitude of visibility is typically between 4° and 10° with a difference of path length through the atmosphere of 6 airmasses. For a typical extinction coefficient of 0·25 magnitudes per airmass, the low Moon suffers 1·5 magnitudes of extra extinction and so the effective threshold should be 1·5 magnitudes brighter. The typical deviation in threshold caused by this effect will be smaller than the typical range just calculated and may be near 0·7 magnitudes.

The final complication for naked eye observing is that, for a given observer on a given night, the sky will darken as the twilight ends so that the threshold will depend on the exact time of observation. The rms variation of the sky brightness for a given Sun–Moon angle will depend on the rms variation for the Sun's depression angle for the ensemble of observations. Since most observations will be made sometime close to the time of best visibility, the total range of variation of the Sun's depression angle will be roughly 2°, which is to say that most of the observations were made within a 16-min interval beginning the same time after sunset (cf. Table II for the naked eye observing interval) for a given lunation. Therefore, the rms variation will be about half the range of variation or 1°. This corresponds typically to a variation by a factor of 3 in the sky brightness (Koomen *et al.* 1952). No extensive data set has been reported in the literature on the variation in the threshold for detecting thin rectangles for various background brightnesses, although the shape of the curves must be similar to the curves for disks as given by Blackwell (1946). In Blackwell's data, a factor of 3 change in the brightness implies a change of typically a factor of 1·7 in the threshold. Hence, the variable sky brightness for a given Sun–Moon angle corresponds to a predicted rms variation of 0·6 magnitudes.

The rms variations of the predicted threshold caused by variations in lunar terrain, observer sensitivity, observer confidence, cusp orientation, lunar altitude, and sky brightness are 0·7, $\leqslant$ 0·75, 0·5, 0·25, 0·7, and 0·6 magnitudes respectively. These variations will all add in quadrature to 1·3 magnitudes as the total rms variation in the threshold. This value is relevant for a data set consisting of observations taken from many different lunations, just as for Danjon's observed variations (see Table III). If all the observations from a data set are taken from one lunation (as with the observed variations from the Moonwatches, see Table III), then the lunar terrain, the cusp orientation, and lunar altitude variations will be small. For such data sets, the variations will add in quadrature to yield a 0·8 magnitude rms change in the predicted threshold.

The predicted rms variation in the arc length can be found from Fig. 2, where the value will be approximately twice the predicted rms variation in the threshold divided by the slope of the average brightness curve evaluated at the threshold. The factor of two is to account for the two cusps. So, for example, the brightness curve for the May Moonwatch (with a Sun–Moon angle of 9°) has a slope of 0·057 magnitudes per degree at the threshold magnitude, so that the predicted rms variation in the arc length will be 28°. The predicted rms variations for the other data sets are given in the last column of Table III.

The correspondence between prediction and observation is close (compare the last two columns of Table III). This agreement provides further support for the validity of the new model.

6 TELESCOPES

The use of telescopes or binoculars is another complication. The detection threshold will change primarily as a function of the increased magnification and light gathering power, since effects due to the transmission of the telescope, the Stiles-Crawford effect, etc, are much smaller and can be ignored (cf. Schaefer 1989). The optics will increase the perceived surface brightness by a factor of $(D/D_e M)^2$, where D is the diameter of the telescope, D_e is the diameter of the pupil, and M is the magnification. If this factor is greater than unity, then light is being wasted by striking the eye outside the pupil such that the factor of increase is unity (Tousey & Hulburt 1948). Therefore, the use of a telescope cannot increase the apparent surface brightness of a resolved source. For typical binoculars, say 7×35s, D is 35 mm, M is 7, and D_e is roughly 5 mm, so that the perceived surface brightness does not change. Generally, telescopic observations of the crescent Moon are made with the lowest power available, which for normal usage is roughly 5× per inch of aperture. With such a magnification, the optics will not greatly change the perceived surface brightnesses.

One of the most important parameters for determining the visibility of an object is the contrast ratio. The contrast ratio is the ratio of the perceived brightness excess towards the object to the perceived brightness of the background. For the crescent Moon, the contrast ratio will be the surface brightness of the Moon divided by the brightness of the surrounding sky. The results from the previous paragraph show that, in practice, the use of optical aid does not change either the perceived surface brightnesses or the contrast ratio.

Another important parameter for determining the visibility of an object is the apparent size. For naked eye observations, the width of the cusps is unresolved whereas with a telescope the cusps may appear wider and resolved. However, Fry (1947) shows that there is only slight dependence of the visibility of long thin rectangles on the width of the rectangle.

The above analysis shows that, in practice, a telescopic and a naked eye observer will view the cusps with the same background sky brightnesses, with the same contrast ratios, and with apparent widths whose difference has no effect. In other words, a telescopic observer (for low magnifications) has no advantage over a naked eye observer for seeing the ends of the cusps. Therefore, the model predictions for arc length with the naked eye (see Fig. 1) are also applicable for (low power) telescopic observers.

A similar analysis can be made for the photographic observations. The emulsion will also be sensitive to some threshold of brightness per unit length, so that the cusps can be detected on the emulsion out to some angle ϕ. However, the problem of sensitivity of photographs is highly complicated, nonlinear, and one which I do not have the expertise to solve. Many different effects must be considered, including scattering within the emulsion, halation, and reciprocity effects. The scattering of photons within the emulsion so as to enlarge any bright source might well contribute to the photographically measured arc lengths larger than 180°. Anyhow, photographic measurements should give arc lengths that behave in a qualitatively similar way to visual measurements.

7 SUMMARY

The brightness per unit length of the lunar crescent falls off steeply towards the cusps. The human eye has some threshold below which it cannot detect a faint cusp, so that some portion of the cusp must be invisible. As the Moon gets closer to the Sun, the crescent gets narrower and dimmer so that less of the cusp is visible. I present a detailed calculation of the lunar brightness (see Fig. 2) based on the definitive Hapke equations which can be used to make specific predictions of the length of the visible arc. For telescopic observers using low magnification (as happens for all crescent observations), the apparent surface brightnesses are essentially the same as if no telescopic aid were used, so that in practice the threshold is nearly the same irrespective of the optics. For a threshold of 8·0 magnitudes, my model predicts that the visible arc length should vary as a function of the Sun–Moon angle as shown in Fig. 1. My model predictions are closely matched both by Danjon's 75 observations and the 70 observations from the two Moonwatches (see Fig. 1). My model also predicts that there will be some scatter around the average curve caused by varying observer sensitivity and other effects. The predicted scatter is close to the observed scatter (see Table III). The close match between the model predictions and the observations gives confidence in the validity of the model.

8 LUNAR CALENDARS

The deduction by Danjon that the crescent will be invisible whenever the Moon is within 7° of the Sun is based on a slight extrapolation of the available observations. As such, for visual observers the Danjon limit is quite solid.

As a theoretical question, the cause of Danjon's limit cannot be: (1) mountain shadows shifting the terminator; (2) the moon's shape being significantly non-spherical; or (3) atmospheric turbulence smearing out the cusps. Instead, I propose that the cusps are not visible because the brightness per unit length is below the eye's detection threshold. This is supported by detailed calculations of the lunar surface brightness using the Hapke equations and by physiological data for the sensitivity of the eye. All available data confirm both the shape of the crescent shortening curve (see Fig. 1) and the scatter about this curve (see Table III). Hence, I conclude that the crescent shortening is merely caused by the faintness of the cusps.

Danjon's limit, regardless of the physical mechanism, can be confidently applied to reject any claimed visual sighting of the crescent whenever the Moon is within 7° of the Sun. However, the converse is not true. That is, just because the Moon is greater than 7° from the Sun does not mean that it is visible. The reason is that other effects (twilight brightness, a low altitude, or a murky atmosphere) also can render the Moon invisible. These other effects enforce a stricter limit on the visibility of the Moon which supersede Danjon's limit (Ilyas 1984b).

Danjon's limit is applicable only to visual observers. Visual observers in aeroplanes or in space will have a similar limit. Electronic or photographic observations may have better or worse detection thresholds (cf. Koomen, Seal & Tousey 1967) which will result in a limit different from the Danjon limit.

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